Data-Driven Control in Infinite-Dimensional Spaces

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Outline

1. Informativity and persistency of excitation
2. Application: Data-driven controllability
3. Application: Data-driven LQR
4. Conclusions and outlook

Two paradigms

Classical (model-based) control

We are given an equation, an input force term, and the space they live in.

For example, the heat equation on $\Omega \subset \mathbb{R}^n$,

$$\partial_t x = \Delta x \ \text{ in } \Omega, \qquad x = u \ \text{ on } \partial\Omega,$$

with $X = L^2(\Omega)$, $A = \Delta$ on $D(A) = H^2 \cap H_0^1$, B the boundary input operator.

Data-driven control

The equation is unknown or partly unknown.
We are given data.

$$(\bar{u}, \bar{y}) = \{(\bar{u}(t), \bar{y}(t)) \ : \ t \in [0, T]\}$$

- $\bar{u}$: the input we choose and apply;
- $\bar{y}$: the output we measure.

1. Informativity and persistency of excitation

Setting: infinite-dimensional systems

$$\begin{cases} \dot{x} = Ax + Bu, & x(0) = x_0, \\ y = Cx, \end{cases}$$

on Hilbert spaces, with $x(t) \in X$, $u(t) \in U$, $y(t) \in Y$, and A the generator of a C_0-semigroup $S(t)$.

The difficulty

For boundary control and point observation, B and C are *unbounded* on X.

The fix (Pritchard and Salamon 1987)

Assume a triple of spaces $W \hookrightarrow X \hookrightarrow V$ makes $B \in \mathcal{L}(U, V)$ and $C \in \mathcal{L}(W, Y)$, together with two admissibility assumptions:

$$\textbf{(H1)} \quad \left\| \int_0^T S(T-s)Bu(s)\,\mathrm{d}s \right\|_W \leq b\|u\|_{L^2(I;U)}, \qquad \textbf{(H2)} \quad \|CS(\cdot)x\|_{L^2(I;Y)} \leq c\|x\|_V.$$

Three examples

1. Heat, Neumann control

$$\begin{cases} \partial_t x = \nu\, \partial_{\xi\xi} x \\ \partial_\xi x(t,0) = u(t) \\ \partial_\xi x(t,1) = 0 \\ y(t) = x(t,\xi_0) \end{cases}$$

analytic, smoothing
B and C unbounded

2. Wave, boundary control

$$\begin{cases} \partial_{tt} w = c^2 \partial_{\xi\xi} w \\ w(t,0) = u(t) \\ w(t,1) = 0 \\ y(t) = \int_0^1 q(\xi) w(t,\xi)\,\mathrm{d}\xi \end{cases}$$

where $x = (w, \partial_t w)$ on $X = L^2(0,1) \times H^{-1}(0,1)$. B unbounded; C bounded

3. Delay equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\big(x(t) - Mx_t\big) = Lx_t + Bu(t) \\ y(t) = Cx_t \end{cases}$$

where $x_t(\cdot) = x(t+\cdot)$. The state $(x(t) - Mx_t, x_t)$ in $X = \mathbb{R}^n \times L^2(-h,0;\mathbb{R}^n)$
B bounded, C unbounded

What makes data “informative”?

Throughout, $(\bar{u}, \bar{x})$ denotes a fixed trajectory: $\bar{x}$ is the mild solution driven by the input $\bar{u} \in L^2(0, T; U)$ from an initial state $\bar{x}_0 \in X$.

Definition 1 (informativity)

$(\bar{u}, \bar{x})$ is *informative* if $\operatorname{span}_{t \in (0,T)} \{(\bar{u}, \bar{x})(t)\}$ is dense in $U \times X$.

Let H be a Hilbert space. For a signal $z \in L^2(0, T; H)$, define the *Gramian operator*

$$G(z) := \int_0^T z(t) \otimes z(t) \, \mathrm{d}t.$$

Definition 2 (informativity, Gramian form)

$(\bar{u}, \bar{x})$ is *informative* if $G(\bar{u}, \bar{x})$ is injective.

- In finite dimensions, injective $\Leftrightarrow$ coercive, i.e., $G(\bar{u}, \bar{x}) \succ 0$ (Schmitz et al. 2024).

How should we choose the input $\bar{u}$ so that $(\bar{u}, \bar{x})$ is informative?

Persistently Excitation in infinite dimensions

Definition

$\bar{u} \in L^2(0, T; U)$ is *persistently exciting (PE) of order L* if, for all $\zeta_0, \ldots, \zeta_{L-1} \in U$,

$$\sum_{k=0}^{L-1} \langle \zeta_k, D^k \bar{u} \rangle_U = 0 \text{ in } \mathscr{D}'(0, T) \quad \Rightarrow \quad \zeta_0 = \cdots = \zeta_{L-1} = 0.$$

It is *PE of infinite order* if it is PE of every finite order.

- When $\bar{u} \in H^{L-1}(0, T; U)$, this is equivalent to injectivity of $G(\bar{u}, \dot{\bar{u}}, \ldots, \bar{u}^{(L-1)})$.

Theorem: $\dim X = N < \infty$ (Willems et al. 2005; Schmitz et al. 2024)

(A, B) is approximately controllable + $\bar{u}$ PE of order $N + 1 \Rightarrow (\bar{u}, \bar{x})$ is informative.

Theorem: $\dim X = \infty$. **We need a stronger notion of PE!**

(A, B) is approximately controllable + $\bar{u}$ PE of infinite order $\not\Rightarrow (\bar{u}, \bar{x})$ informative.

Harmonic persistency of excitation

Informal definition

A harmonic signal $h(t) = \sum_\ell v_\ell \mathrm{e}^{i\omega_\ell t}$ is *harmonically PE* if there is a total family (ϕ_j) in U such that each ϕ_j is excited at infinitely many frequencies accumulating at some $\nu_j \in \mathbb{R}$.

Lemma

Harmonic PE $\Rightarrow$ PE of infinite order.

Theorem (Approximate controllability characterization)[1]

Assume A generates an *analytic* semigroup and let $\bar{u}(t) = \mathrm{e}^{\sigma t} h(t)$. Then

$$(A, B) \text{ approximately controllable} \quad \Longleftrightarrow \quad (\bar{u}, \bar{x}) \text{ informative.}$$

[1] $B^\star$ bounded on $\mathcal{D}(A^\star)$, transfer map $(s - A_{-1})^{-1}B$ decaying, σ beyond the growth bound of S.

2. Application: Data-driven controllability

A data-driven controllability test

Fattorini–Hautus Theorem (Boyer 2020, Thm. III.3.7)[2]

(A, B) is approximately controllable $\iff$ $\ker B^\star \cap \ker(A^\star - \lambda) = \{0\}$ for all $\lambda \in \mathbb{C}$.

Lemma: controllability certificate (input–state).

If for all $\eta \in X \setminus \{0\}$ and $\lambda, \kappa \in \mathbb{C}$,

$$\langle \eta, \bar{x}(t) \rangle \neq \kappa \, \mathrm{e}^{\lambda t} \text{ on } (0, T)$$

then (A, B) is approximately controllable.

Lemma: controllability certificate (input–output).

Let (A, C) be observable in time T. If for all $v \in L^2(0, T; U_\mathbb{C})$, $g \in L^2(0, T; Y_\mathbb{C}) \setminus \{0\}$, $\lambda, \kappa \in \mathbb{C}$,

$$\int_0^T \left(\langle v(t), \bar{u}(s+t) \rangle_U + \langle g(t), \bar{y}(s+t) \rangle_Y \right) \mathrm{d}t \neq \kappa \, \mathrm{e}^{\lambda s} \text{ on } (0, T')$$

then (A, B) is approximately controllable.

[2] A analytic, $A^\star$ with compact resolvent and complete root vectors, $B^\star$ bounded on $\mathcal{D}(A^\star)$.

Controllability from one record

Data read

i-s: input–state

i-s-o: input–state–output

i-o: input–output

✓ / ✗

controllable / not controllable

Configuration		data	found	recovered λ	exact λ
heat, one-sided	✓	i-s	0	—	—
		i-o	0	—	—
heat, symmetric	✗	i-s	1	-0.783	-0.790
		i-o	1	-0.783	-0.790
wave, one-sided	✓	i-s	0	—	—
		i-o	0	—	—
wave, symmetric	✗	i-s	2	$0 \pm 3.129\mathrm{i}$	$0 \pm 3.142\mathrm{i}$
		i-o	2	$0 \pm 3.128\mathrm{i}$	$0 \pm 3.142\mathrm{i}$
delay, coupled	✓	i-s	0	—	—
		i-o	0	—	—
delay, decoupled	✗	i-s	13	$-1.102 \pm 1.051\mathrm{i}$	$-1.106 \pm 1.046\mathrm{i}$
		i-o	2	$-1.101 \pm 1.048\mathrm{i}$	$-1.106 \pm 1.046\mathrm{i}$

3. Application: Data-driven LQR

Willems' fundamental lemma

Input–output behavior on the horizon T:

$$\mathcal{B}_T := \{(u, y) \in L^2(0, T; U \times Y) : \; y = Cx, \; x \text{ the mild solution of } \dot{x} = Ax + Bu, \; x(0) \in X\}.$$

One record $(\bar{u}, \bar{y})$ on a horizon of length $T + T'$.

Definition.

$(\bar{u}, \bar{x})$ is *window informative* if

$$\overline{\operatorname{im} \mathcal{Z}_T} = L^2(0, T; U) \times X,$$

where, for $\phi \in L^2(0, T')$,

$$\mathcal{Z}_T \phi := \int_0^{T'} \phi(s) \left(\bar{u}(s + \cdot), \bar{x}(s) \right) \mathrm{d}s.$$

Lemma. Window informativity $\Rightarrow$ informativity. The converse holds for harmonic PE inputs.

Willems' fundamental lemma.

Let $(\bar{u}, \bar{x})$ be window informative. Then

$$\overline{\operatorname{im} \mathcal{Y}_T} = \overline{\mathcal{B}_T},$$

where, for $\phi \in L^2(0, T')$,

$$\mathcal{Y}_T \phi := \int_0^{T'} \phi(s) \left(\bar{u}(s + \cdot), \bar{y}(s + \cdot) \right) \mathrm{d}s.$$

- (A, C) approximately observable in time T: every $(u, y) \in \mathcal{B}_T$ determines its latent state x.
- (A, C) exactly observable: $\mathcal{B}_T$ is closed.

Data-driven LQR

Model-based LQR (Pritchard and Salamon 1987): minimize

$$J(u; x_0) = \int_0^T \left(\|y\|_Y^2 + \langle u, Ru\rangle_U\right) \mathrm{d}t, \qquad u_F = -R^{-1}B^\star P(t)x,$$

where P solves a Riccati equation.

Theorem: Data-driven regulator

With windows of length $T_0 + T$, $(\bar{u}, \bar{x})$ window informative, and (A, C) observable in time T_0,

$$\min\ \mathcal{J}(u, y) \quad \text{s.t.} \quad (u, y) \in \overline{\operatorname{im} \mathcal{Y}_{T_0+T}}, \quad (u, y)|_{[-T_0, 0]} = (u_{\text{ini}}, y_{\text{ini}})$$

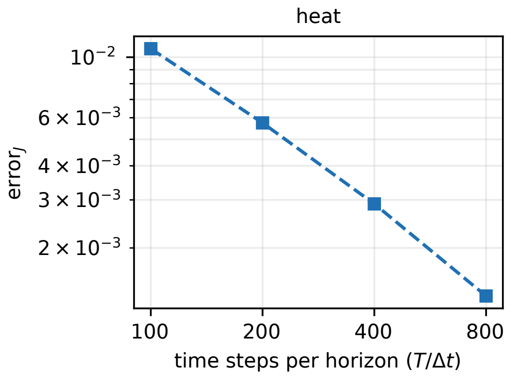
has the *same* minimizer as Riccati.

Data-driven LQR: input–output data

Example	J	error_J
heat	0.3154	0.005745
wave	0.8624	0.004255
delay	0.06126	0.01754

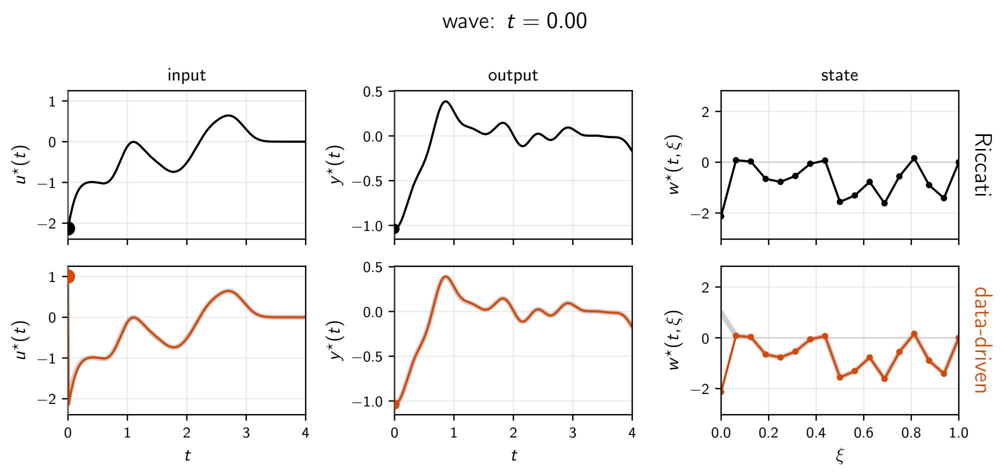
J: cost of the data-driven optimum;
$J_\star$: Riccati optimum,

$$\mathrm{error}_J = \frac{|J - J_\star|}{J_\star}.$$



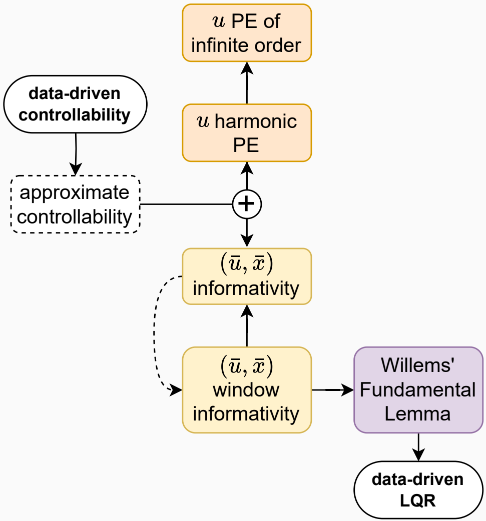
LQR: Model-based vs. Data-driven

$$\begin{cases} \partial_{tt} w = c^2 \partial_{\xi\xi} w \\ w(t,0) = u(t), \quad w(t,1) = 0 \\ y(t) = \int_0^1 q(\xi) w(t,\xi)\,\mathrm{d}\xi \end{cases} \qquad J(u; x_0) = \int_0^4 \left(\|y\|_Y^2 + \langle u, Ru \rangle_U\right) \mathrm{d}t, \quad R = 0.05.$$



4. Conclusions and outlook

Conclusions and outlook



Open problems:

- Convergence and stability of the **numerical schemes**.
- **The rest of the informativity catalogue**: stabilizability, dissipativity.
- **Best choice of input**: how to design $\bar{u}$ to maximize the information content of $(\bar{u}, \bar{x})$.

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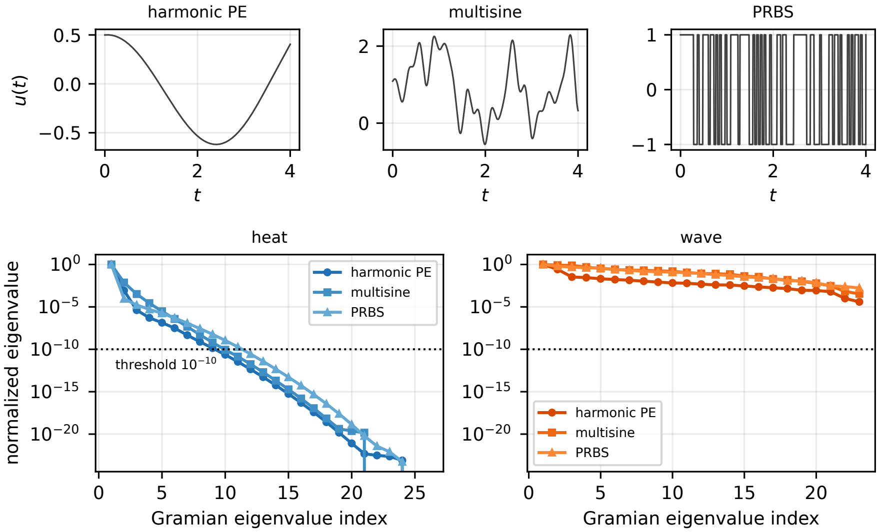
Excitation and conditioning



5. Harmonic PE, formally

Harmonic signals and harmonic PE

Definition (Harmonic signal)

$\bar{u} \in L^2(I; U)$ is *harmonic* if it is the restriction to I of an almost-periodic signal

$$\bar{u}(t) = \sum_{\ell \geq 1} v_\ell \, \mathrm{e}^{i\omega_\ell t}, \qquad v_\ell \in U_{\mathbb{C}},$$

with (ω_ℓ) pairwise distinct, $\sup_\ell |\omega_\ell| < \infty$ and $\sum_{\ell \geq 1} \|v_\ell\|_U < \infty$, so that $\bar{u}(t) \in U$ for real t.

Definition (Harmonically persistently exciting)

A harmonic signal h is *harmonically PE* if there is a total family $(\phi_j)_{j\geq 1} \subset U_{\mathbb{C}}$, pairwise distinct indices $\ell(j,k)$, $j,k \geq 1$, nonzero scalars $a_{j,k}$ and a number $\nu_j \in \mathbb{R}$ such that

$$v_{\ell(j,k)} = a_{j,k}\phi_j \quad \text{and} \quad \omega_{\ell(j,k)} \xrightarrow{k\to\infty} \nu_j.$$

Example: harmonically PE signal For a real orthonormal basis $(\phi_j)_{j\geq 1}$ of U,

$$h(t) = \sum_{j,k\geq 1} 2^{-j-k} \cos(\omega_{j,k} t)\, \phi_j, \qquad \omega_{j,k} = 2 - 2^{-j} - 2^{-j-k}.$$

A boundary-controlled example

Example (Neumann heat equation, $\nu = 1$)

Take $\sigma = 0$ and

$$\bar{u}(t) := \sum_{k \geq 1} 2^{-k} \cos(\omega_k t), \qquad \omega_k := 1 + \frac{1}{k+1}.$$

Then $\bar{u}$ is harmonically PE and every hypothesis of the sufficiency theorem holds — although B is *unbounded*. Consequently, for every $x_0 \in L^2(0,1)$ the record $(\bar{u}, \bar{x})$ is informative on I, and state-window informative on $[0, T + T']$ for every $T' > 0$.

- The semigroup is bounded but not exponentially stable ($\omega_0 = 0$): the frequencies are kept away from the spectrum instead of the exponent being pushed past it.
- Wave and delay semigroups are not analytic — still open.